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The conformal anomaly of string theory derived from the path integral approach

Wei Chen†‡, Guang-jiong Ni†§ and Rong-tai Wang§

† Center of Theoretical Physics, CCAST (World Laboratory), Beijing, People's Republic of China

‡ Institute of High Energy Physics, Academia Sinica, Beijing, People's Republic of China

§ Modern Physics Institute, Fudan University, Shanghai, People's Republic of China

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Abstract. Using a comoving representation, we derive the conformal anomaly of string theory in an alternative path integral formulation. Accordingly, the critical dimensions for bosonic and spinning strings are obtained by the residue regularisation method. The possibility of constructing a consistent quantum string theory under the subcritical dimension is also discussed.

1. Introduction

It is well known that the classical bosonic string can be described by the action (Polyakov 1981)

$$S=-\tfrac{1}{2}\int \mathrm{d}^2\sigma \sqrt{g}g^{ab}\,\partial_a X^\mu\,\partial_b X_\mu \tag{1.1}$$

which has two local symmetries, the reparametrisation and the Weyl rescaling. As a consequence of the local Weyl rescaling invariance, the classical energy-momentum tensor is traceless. However, the quantisation would in general violate this invariance—the conformal anomaly, also called the trace anomaly, appears. While Polyakov (1981) first pointed out in the path integral scheme that the conformal anomaly cancellation results in a critical dimension $D=26$ for the base space, Fujikawa (1979, 1980, 1982, 1983) rederived this result in a manifestly BRST-invariant manner, using the heat kernel (or Gaussian) regularisation scheme that he proposed in the path integral approach to the chiral anomaly. Recently, many authors have continued to attack this problem (Bouwknegt and van Nieuwenhuizen 1986, Alvarez 1987, Petcher and Van Holten 1987). In this paper, we wish to re-examine the conformal anomaly of string theory using an alternative formulation of the path integral method proposed by two of us (Wang and Ni (1987), hereafter referred to as I; see also Wang and Ni (1988)).

The organisation of this paper is as follows. In § 2 the generating functional of the bosonic string is treated in a comoving representation (I). In § 3 the anomalous action is calculated by the residue regularisation method (I) which results in the critical dimension $D=26$. Similar manipulation for a spinning string is presented in § 4. Section 5 contains a summary and discussion.

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2. The bosonic string

We begin from the classical action of the bosonic string (1.1). First of all, a Wick rotation brings us to Euclidean space on the world sheet and a conformal gauge is chosen as usual

$$g_{ab}(\sigma)=\rho(\sigma)\delta_{ab} \qquad a, b=1,2. \tag{2.1}$$

A BRST invariance could be realised by introducing the Faddeev–Popov ghosts $\eta(\sigma)=\binom{\eta^1(\sigma)}{\eta^2(\sigma)}$ and antighosts $\xi(\sigma)=(\xi_1(\sigma), \xi_2(\sigma))$. They help to fix the gauge to $g_{12}(\sigma)=0$ and $g_{11}(\sigma)-g_{22}(\sigma)=0$. Then the quantum bosonic string is characterised by the generating functional (Fujikawa 1982)

$$\begin{aligned} Z&=\int\prod_\sigma[\mathrm{d}C][\mathrm{d}\tilde{X}^\mu][\mathrm{d}\xi][\mathrm{d}\tilde{\eta}]\,\mathrm{e}^{-S}\\ &=\int\prod_\sigma[\mathrm{d}C]Z[\rho] \end{aligned}\tag{2.2}$$

$$S=\tfrac{1}{2}\int\mathrm{d}^2\sigma\,\partial_a\left(\frac{\tilde{X}^\mu}{\sqrt{\rho}}\right)\partial_a\left(\frac{\tilde{X}_\mu}{\sqrt{\rho}}\right)+\mathrm{i}\int\mathrm{d}^2\sigma\,\xi\sqrt{\rho}\,\bar{\partial}\left(\frac{1}{\rho}\tilde{\eta}\right)\tag{2.3}$$

where $|C|=\sqrt{\rho}$ and $\bar{\partial}=\tau_1\partial_1+\tau_3\partial_2$ with τ_1 and τ_3 being Pauli matrices, while $\tilde{X}_\mu=\sqrt{\rho}X_\mu$ and $\tilde{\eta}=\rho\eta$ are treated as the independent variables. After the Faddeev–Popov prescription, the gauge-fixed action (2.3) still preserves the local Weyl invariance under the rescaling transformations

$$\begin{aligned} &\rho(\sigma)\to\exp[\alpha(\sigma)]\rho(\sigma) && \tilde{X}^\mu(\sigma)\to\exp[\tfrac{1}{2}\alpha(\sigma)]\tilde{X}^\mu(\sigma)\\ &\xi(\sigma)\to\exp[-\tfrac{1}{2}\alpha(\sigma)]\xi(\sigma) && \tilde{\eta}(\sigma)\to\exp[\alpha(\sigma)]\tilde{\eta}(\sigma). \end{aligned}\tag{2.4}$$

For a finite $\alpha(\sigma)$, let us introduce a comoving representation and divide the transformation into N steps: $\alpha_i(\sigma)=t_i\alpha(\sigma)$, $t_0=0,\ldots,t_N=1$. Defining $\tilde{X}^{\mu(i)}=\mathrm{e}^{\alpha_i/2}\tilde{X}^\mu$, $\xi^{(i)}=\mathrm{e}^{-\alpha_i/2}\xi$ and $\tilde{\eta}^{(i)}=\mathrm{e}^{\alpha_i}\tilde{\eta}$, one can write at any intermediate step

$$\begin{aligned} Z[\rho\,\mathrm{e}^{\alpha_i}]&=\int[\mathrm{d}\tilde{X}^{\mu(i)}][\mathrm{d}\xi^{(i)}][\mathrm{d}\tilde{\eta}^{(i)}]\\ &\quad\times\exp\Big(-\int\mathrm{d}^2\sigma(-\tfrac{1}{2}\tilde{X}^{\mu(i)}\,\mathrm{e}^{-\alpha_i/2}\rho^{-1/2}\partial_a\partial_a\rho^{-1/2}\,\mathrm{e}^{-\alpha_i/2}\tilde{X}^{(i)}_\mu\\ &\quad+\mathrm{i}\xi^{(i)}\,\mathrm{e}^{\alpha_i/2}\rho^{1/2}\bar{\partial}\rho^{-1}\,\mathrm{e}^{-\alpha_i}\tilde{\eta}^{(i)})\Big)\\ &=\int[\mathrm{d}\tilde{X}^\mu][\mathrm{d}\xi][\mathrm{d}\tilde{\eta}]\exp\Big(-\int\mathrm{d}^2\sigma(\tfrac{1}{4}\tilde{X}^\mu R_0^{(i)\dagger}R_0^{(i)}\tilde{X}_\mu+\xi R_1^{(i)}\tilde{\eta})\Big) \end{aligned}\tag{2.5}$$

where an operator

$$R_n^{(i)}=\mathrm{i}\exp[\tfrac{1}{2}n\alpha_i]\rho^{n/2}\bar{\partial}\rho^{-(n+1)/2}\exp[-\tfrac{1}{2}(n+1)\alpha_i]\tag{2.6}$$

is defined in general with n being an integer or half integer. Notice that

$$R_n^{(i)\dagger}=R_{-n-1}^{(i)}\tag{2.7}$$

so $R_n^{(i)}$ is not a Hermitian operator except for $n=-\frac{1}{2}$. However, both $R_n^{(i)\dagger}R_n^{(i)}$ and $R_n^{(i)}R_n^{(i)\dagger}$ are Hermitian. Thus there are two component functions $u_{ns}^{(i)}$ and $v_{ns}^{(i)}$ satisfying (I)

$$\begin{aligned} R_n^{(i)}u_{ns}^{(i)} &= \lambda_{ns}^{(i)}\exp(\mathrm{i}\vartheta_{ns}^{(i)})v_{ns}^{(i)} \\ R_n^{(i)\dagger}v_{ns}^{(i)} &= \lambda_{ns}^{(i)}\exp(-\mathrm{i}\vartheta_{ns}^{(i)})u_{ns}^{(i)} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} R_n^{(i)\dagger}R_n^{(i)}u_{ns}^{(i)} &= \lambda_{ns}^{(i)2}u_{ns}^{(i)} \\ R_n^{(i)}R_n^{(i)\dagger}v_{ns}^{(i)} &= \lambda_{ns}^{(i)2}v_{ns}^{(i)} \end{aligned} \tag{2.9}$$

where $\{u_{ns}^{(i)}\}$ and $\{v_{ns}^{(i)}\}$ $(s=0,1,2,\ldots)$ each constitute a complete orthonormal set. The extra factor $\frac{1}{2}$ in the first term of the integrand in (2.5) is due to the fact that $R_0^\dagger R_0=-\rho^{-1/2}\bar{\eth}\eth\rho^{-1/2}$ is a 2×2 matrix operator whereas $-(1/\sqrt{\rho})\partial_a\partial_a(1/\sqrt{\rho})$ is a scalar Laplacian.

As in I, one can write

$$\begin{aligned} Z[\rho] &= \frac{Z[\rho\,\mathrm{e}^{\alpha_0}]}{Z[\rho\,\mathrm{e}^{\alpha_1}]}\frac{Z[\rho\,\mathrm{e}^{\alpha_1}]}{Z[\rho\,\mathrm{e}^{\alpha_2}]}\cdots\frac{Z[\rho\,\mathrm{e}^{\alpha_{N-1}}]}{Z[\rho\,\mathrm{e}^{\alpha_N}]}Z[\rho\,\mathrm{e}^{\alpha}] \\ &= \exp[\Delta\Gamma(\rho,\alpha)]Z[\rho\,\mathrm{e}^{\alpha}] \end{aligned} \tag{2.10}$$

where

$$\exp[\Delta\Gamma(\rho,\alpha)] = \prod_{i=0}^{N-1}\frac{Z[\rho\,\mathrm{e}^{\alpha t_i}]}{Z[\rho\,\mathrm{e}^{\alpha t_{i+1}}]} \tag{2.11}$$

$$\Delta\Gamma = \sum_{i=0}^{N-1}\delta\Gamma_i \tag{2.11$'$}$$

$$\begin{aligned} \delta\Gamma_i &= -\tfrac{1}{2}D\ln\det(R_0^{(i)\dagger}R_0^{(i)})+\ln\det(R_1^{(i)})-[-\tfrac{1}{2}D\ln\det(R_0^{(i+1)\dagger}R_0^{(i+1)})+\ln\det(R_1^{(i+1)})] \\ &= -\tfrac{1}{2}D\,\mathrm{Tr}\ln(R_0^{(i)\dagger}R_0^{(i)})+\mathrm{Tr}\ln(R_1^{(i)}) \\ &\quad +\tfrac{1}{2}D\,\mathrm{Tr}\ln(R_0^{(i+1)\dagger}R_0^{(i+1)})-\mathrm{Tr}\ln(R_1^{(i+1)}). \end{aligned} \tag{2.12}$$

As $N\to\infty$, $\delta t_i=t_{i+1}-t_i\to 0$ and one has

$$\delta(R_0^{(i)\dagger}R_0^{(i)}) = R_0^{(i+1)\dagger}R_0^{(i+1)}-R_0^{(i)\dagger}R_0^{(i)} = \{-\tfrac{1}{2}\alpha_i(\sigma),R_0^{(i)\dagger}R_0^{(i)}\}\delta t_i \tag{2.13}$$

$$\delta R_1^{(i)} = R_1^{(i+1)}-R_1^{(i)} = (\tfrac{1}{2}\alpha_i(\sigma)R_1^{(i)}-R_1^{(i)}\alpha_i(\sigma))\delta t_i. \tag{2.14}$$

Then, e.g.,

$$\begin{aligned} &\mathrm{Tr}\ln R_1^{(i+1)}-\mathrm{Tr}\ln R_1^{(i)} \\ &\quad = \sum_s\int \mathrm{d}^2\sigma\, v_{1s}^{\dagger(i)}\ln\left(\frac{R_1^{(i)}+\delta R_1^{(i)}}{R_1^{(i)}}\right)u_{1s}^{(i)} \\ &\quad = \left(\tfrac{1}{2}\sum_s\int \mathrm{d}^2\sigma\, v_{1s}^{\dagger(i)}\alpha_i(\sigma)v_{1s}^{(i)}-\sum_s\int \mathrm{d}^2\sigma\, u_{1s}^{\dagger(i)}\alpha_i(\sigma)u_{1s}^{(i)}\right)\delta t_i \end{aligned} \tag{2.15}$$

where equations (2.14) and (2.8) and the approximation $\ln(1+x)\simeq x$ have been used. Therefore

$$\delta\Gamma_i = \delta t_i\int \mathrm{d}^2\sigma\,\alpha_i(\sigma)\left(-\tfrac{1}{4}D\sum_s u_{0s}^{\dagger(i)}u_{0s}^{(i)}-\tfrac{1}{2}\sum_s v_{1s}^{(i)\dagger}v_{1s}^{(i)}+\sum_s u_{1s}^{(i)\dagger}u_{1s}^{(i)}\right). \tag{2.16}$$

3. The method of residue regularisation

Obviously, each of the three sums in (2.16) is divergent, so a regularisation procedure is needed. Hence, instead of the ill-defined trace

$$\mathrm{Tr}({}^{(n)}\alpha_i)=\int \mathrm{d}^2\sigma\, \alpha_i(\sigma)\sum_s u_{ns}^{(i)\dagger}(\sigma)u_{ns}^{(i)}(\sigma) \tag{3.1}$$

we define a regularised trace

$$[\mathrm{Tr}({}^{(n)}\alpha_i)]_{\mathrm{reg}}=\left(\int \mathrm{d}^2\sigma\, \alpha_i(\sigma)\sum_s u_{ns}^{(i)\dagger}f(\xi, R_n^{(i)\dagger}R_n^{(i)})u_{ns}\right)_{\xi\text{-independent}} \tag{3.2}$$

where

$$f(\xi, R_n^{(i)\dagger}R_n^{(i)})=\frac{1}{2\pi \mathrm{i}}\oint_C \frac{\mathrm{e}^{-\xi z}}{z-R_n^{(i)\dagger}R_n^{(i)}}\,\mathrm{d}z. \tag{3.3}$$

The contour C on the complex z plane is selected such that it bypasses the origin and ends at Re $z\to\infty$ (see I). In what follows we shall trade the summation over eigenstates of $R_n^{(i)\dagger}R_n^{(i)}$ with the integration in momentum space. Thus in the space spanned by the plane wave $\exp(\mathrm{i}k_\mu\sigma_\mu)$, if we denote

$$\rho=\mathrm{e}^{\phi} \qquad \tilde{\phi}=\phi+\alpha_i \tag{3.4}$$

then the operators $R_n^{(i)}$ and $R_n^{(i)\dagger}$ can be recast into

$$\begin{aligned} R_n^{(i)} &= Q_n - \not{k}\,\mathrm{e}^{-\tilde{\phi}/2} \\ R_n^{(i)\dagger} &= Q_n^\dagger - \not{k}\,\mathrm{e}^{-\tilde{\phi}/2} \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} Q_n &= \mathrm{i}\,\mathrm{e}^{-\tilde{\phi}/2}\not{d} - \mathrm{i}\tfrac{1}{2}(n+1)\,\mathrm{e}^{-\tilde{\phi}/2}(\not{d}\tilde{\phi}) \\ Q_n^\dagger &= \mathrm{i}\,\mathrm{e}^{-\tilde{\phi}/2}\not{d} + \mathrm{i}\tfrac{1}{2}n\,\mathrm{e}^{-\tilde{\phi}/2}(\not{d}\tilde{\phi}) \end{aligned} \tag{3.6}$$

with the derivative symbol $\not{d}$ only acting on $\tilde{\phi}(\sigma)$ situated to its right. So

$$Q_n^\dagger Q_n = \mathrm{e}^{-\tilde{\phi}}[\tfrac{1}{2}(n+1)\tfrac{1}{2}(n-1)(\not{d}\tilde{\phi})^2+\tfrac{1}{2}(n+1)(\not{d}^2\tilde{\phi})+(\not{d}\tilde{\phi})\not{d}-\not{d}\not{d}] \tag{3.7}$$

$$R_n^{(i)\dagger}R_n^{(i)} = Q_n^\dagger Q_n + k^2\,\mathrm{e}^{-\tilde{\phi}} - K \tag{3.8}$$

where

$$K=\mathrm{i}\,\mathrm{e}^{-\tilde{\phi}}(\not{k}\not{d}+\not{d}\not{k}-\tfrac{1}{2}(n+1)\not{k}(\not{d}\tilde{\phi})+\tfrac{1}{2}(n-1)(\not{d}\tilde{\phi})\not{k}). \tag{3.9}$$

Furthermore, an expansion

$$\frac{1}{z-R_n^{(i)\dagger}R_n^{(i)}}=\sum_{l=0}^{\infty}\frac{(-1)^l}{(z-k^2\,\mathrm{e}^{-\tilde{\phi}}-Q_n^\dagger Q_n)^{l+1}}K^l \tag{3.10}$$

is used without worrying about its convergence property. This is because when we pick the contributions of residues in integral (3.3) on the complex z plane, only the ξ-independent terms are taken into account, so only finite terms in (3.10) survive. The integration with respect to k is performed by using a well known formula

$$\int_0^\infty \frac{\mathrm{d}(k^2)\,(k^2)^{m_1}}{(z-k^2)^{m_2}}=\frac{(-1)^{m_1+1}}{z^{m_2-(m_1+1)}}B(m_2-(m_1+1), m_1+1) \tag{3.11}$$

where $B(\mu,\nu)=\Gamma(\mu)\Gamma(\nu)/\Gamma(\mu+\nu)$ is the beta function.

After careful calculation, we find

$$[\mathrm{Tr}({}^{(n)}\alpha_i)]_{\mathrm{reg}} = \int \mathrm{d}^2\sigma\, \alpha_i(\sigma)\left(-\frac{3n+1}{24\pi}\right) \mathrm{Tr}\, \not{d}^2\tilde{\phi}$$

$$= -\frac{3n+1}{12\pi}\int \mathrm{d}^2\sigma\, t_i\alpha(\sigma)[\partial^2 \ln\rho + t_i\partial^2\alpha]. \tag{3.12}$$

In order to regularise the summation $\Sigma_s\, v_{ns}^{\dagger(i)} v_{ns}^{(i)}$, we should use $R_n^{(i)}R_n^{(i)\dagger}$ to replace $R_n^{(i)\dagger}R_n^{(i)}$ in (3.3). Thanks to (2.7), $R_n^{(i)}R_n^{(i)\dagger} = R_{-n-1}^{(i)\dagger}R_{-n-1}^{(i)}$ and no extra labour is needed. It is now easy to evaluate $\delta\Gamma_i$ by simply substituting (3.12) for $n = 0$, -2 and 1 into (2.16). This gives

$$\delta\Gamma_i = -\delta t_i \frac{D-26}{48\pi}\int \mathrm{d}^2\sigma\, t_i\alpha(\sigma)(\partial^2\ln\rho + t_i\partial^2\alpha). \tag{3.13}$$

To calculate the anomalous action (2.11′), we turn to continuous t ($t_i \to t$, $\delta t_i \to \mathrm{d}t$, $\delta\Gamma_i \to \mathrm{d}\Gamma$):

$$\Delta\Gamma = \int_{t=0}^{1} \mathrm{d}\Gamma = -\frac{D-26}{72\pi}\int \mathrm{d}^2\sigma\, \alpha(\sigma)[3\partial^2\ln\rho + 2\partial^2\alpha]. \tag{3.14}$$

Going back to (2.10), one has

$$\begin{aligned} Z[\rho] = \mathrm{e}^{\Delta\Gamma}\int [\mathrm{d}\tilde{X}^\mu][\mathrm{d}\xi][\mathrm{d}\tilde{\eta}] \exp\Big\{-\int \mathrm{d}^2\sigma\Big[&-\frac{1}{2}\mathrm{e}^{-\alpha}\frac{\tilde{X}^\mu}{\sqrt{\rho}}\partial_a\partial_a\frac{\tilde{X}_\mu}{\sqrt{\rho}} + \frac{1}{2}\mathrm{e}^{-\alpha}(\partial_a\alpha)\frac{\tilde{X}^\mu}{\sqrt{\rho}}\partial_a\frac{\tilde{X}_\mu}{\sqrt{\rho}} \\ &+\frac{1}{4}\mathrm{e}^{-\alpha}\frac{1}{\rho}(\partial_a\partial_a\alpha)\tilde{X}^\mu\tilde{X}_\mu - \frac{1}{8}\mathrm{e}^{-\alpha}\frac{1}{\rho}(\partial_a\alpha)(\partial_a\alpha)\tilde{X}^\mu\tilde{X}_\mu \\ &+\mathrm{i}\,\mathrm{e}^{-\alpha/2}\xi\sqrt{\rho}\left(\bar{\partial}\frac{1}{\rho}\tilde{\eta}\right) - \mathrm{i}\,\mathrm{e}^{-\alpha/2}\xi\sqrt{\rho}(\bar{\partial}\alpha)\frac{1}{\rho}\tilde{\eta}\Big]\Big\}. \end{aligned} \tag{3.15}$$

Differentiating (3.15) with respect to α and then letting $\alpha \to 0$, one obtains from

$$\left.\frac{\delta Z[\rho]}{\delta\alpha(\sigma)}\right|_{\alpha\to 0} = 0 \tag{3.16}$$

the following equation:

$$\left\langle\left[-\frac{1}{2}\frac{\tilde{X}^\mu}{\sqrt{\rho}}\partial_a\partial_a\frac{\tilde{X}_\mu}{\sqrt{\rho}} - \mathrm{i}\xi\sqrt{\rho}\left(\bar{\partial} + \frac{1}{2}\vec{\partial}\right)\frac{1}{\rho}\tilde{\eta}\right]\right\rangle = \frac{(D-26)}{24\pi}\partial^2\ln\rho \tag{3.17}$$

where $\langle\ldots\rangle$ denotes the quantum average. Equation (3.17) implies that the energy-momentum tensor, which is conserved at the classical level due to the Weyl invariance, now acquires an anomalous contribution at the quantum level

$$\langle T_{11} + T_{22}\rangle = \frac{(D-26)}{24\pi}\partial^2\ln\rho. \tag{3.18}$$

Evidently, in the critical dimension of base space $D = 26$, one can construct a consistent quantum bosonic string without a conformal anomaly. This fact may be seen better in (2.10) [i.e. (3.15)] with $Z[\rho\,\mathrm{e}^\alpha]$ shown as in (2.5). It is the non-invariance of the functional integration measure under the transformations (2.4) which is responsible for the appearance of the anomalous action, $\Delta\Gamma$; the latter is obviously not invariant under $\rho \to \rho\exp[\beta(\sigma)]$. The possibility of constructing a consistent quantum bosonic string in a subcritical dimension will be discussed in the final section.

4. The spinning string

We now turn to the spinning string (Brink *et al* 1976, 1977, Deser and Zumino 1976). This is a two-dimensional supergravity. When performing the quantisation, besides the reparametrisation symmetry, the local supersymmetry gauge must also be fixed. According to the Faddeev–Popov trick, the bosonic ghosts for the gravitino have to be introduced. In the superconformal gauge, the full quantum Lagrangian density is (Bouwknegt and van Nieuwenhuizen 1986)

$$\mathscr{L}=\tfrac{1}{2}\partial_a\left(\frac{\tilde{X}^{\mu}}{\sqrt{\rho}}\right)\partial_a\left(\frac{\tilde{X}_{\mu}}{\sqrt{\rho}}\right)+\tfrac{1}{2}\mathrm{i}\bar{\tilde{\psi}}\rho^{-1/4}\partial\!\!\!/\,\rho^{-1/4}\tilde{\psi}_{\mu}+\mathrm{i}\xi\sqrt{\rho}\,\partial\!\!\!/\,\frac{1}{\rho}\,\tilde{\eta}+\mathrm{i}\bar{\tilde{\lambda}}\rho^{1/4}\partial\!\!\!/\,\rho^{-3/4}\tilde{\zeta} \tag{4.1}$$

where the fermion coordinates, the bosonic ghost and antighost have also been redefined as $\tilde{\psi}_{\mu}=\rho^{1/4}\psi_{\mu}$, $\tilde{\zeta}=\rho^{3/4}\zeta$ and $\tilde{\lambda}=\rho^{-1/4}\lambda$, respectively. As in § 2 for the bosonic string case, the expression (4.1), and thereby the action, are invariant under the rescaling transformations (2.4) together with the following:

$$\tilde{\psi}^{\mu}\to \mathrm{e}^{\alpha/4}\tilde{\psi}^{\mu}\qquad \tilde{\zeta}\to \mathrm{e}^{3\alpha/4}\zeta\qquad \tilde{\lambda}\to \mathrm{e}^{-\alpha/4}\tilde{\lambda}. \tag{4.2}$$

Repeating steps similar to those leading to (2.16), for the anomalous action acquired by an infinitesimal transformation stemming from $\alpha_i(\sigma)=t_i\alpha(\sigma)$ to $(t_i+\delta t_i)\alpha(\sigma)$, we now obtain

$$\begin{aligned}\delta\Gamma_i=\delta t_i\int \mathrm{d}^2\sigma\,\alpha_i(\sigma)\Big(&-\tfrac{1}{4}D\sum_s u_{0s}^{(i)\dagger}u_{0s}^{(i)}+\tfrac{1}{4}D\sum_s u_{-1/2,s}^{(i)\dagger}u_{-1/2,s}^{(i)}-\tfrac{1}{2}\sum_s v_{1s}^{(i)\dagger}v_{1s}^{(i)}\\&+\sum_s u_{1s}^{(i)\dagger}u_{1s}^{(i)}+\tfrac{1}{4}\sum_s v_{-1/2,s}^{(i)\dagger}v_{-1/2,s}^{(i)}-\tfrac{3}{4}\sum_s u_{-1/2,s}^{(i)\dagger}u_{-1/2,s}^{(i)}\Big).\end{aligned} \tag{4.3}$$

Substituting (3.12) for $n=0,\ -\tfrac{1}{2},\ -2,\ 1,\ -\tfrac{3}{2}$ and $\tfrac{1}{2}$ into (4.2), one gets

$$\delta\Gamma_i=-\delta t_i\frac{(D-10)}{32\pi}\int \mathrm{d}^2\sigma\,t_i\alpha(\sigma)(\partial^2\ln\rho+t_i\partial^2\alpha). \tag{4.4}$$

Thus the critical dimension $D=10$ follows immediately.

5. Summary and discussion

Though it had been speculated for years that the theory of superstrings may be a candidate for the unified model for all known interactions, i.e. the theory of everything, a series of cardinal problems have to be answered before such a model can be put on a sound basis. Among them the problem of the conformal anomaly with relevant critical dimension occupies an important position. It deserves to be investigated from different aspects and by different approaches. What we have done in this paper is to re-examine this problem by the path integral method while using the comoving representation together with a residue regularisation scheme (I). We wish to make two remarks.

As discussed in I, the residue regularisation method used for the chiral anomaly may be understood as choosing the zero-mode contributions of a non-Hermitian operator ($\mathrm{i}D\!\!\!\!/\,$). The situation for the conformal anomaly considered here seems not to be so simple. Let us formally compute the difference between two Tr α, one regularised by $R_n^\dagger R_n$ and the other by $R_n R_n^\dagger$, and set $\alpha = 1$:

$$\operatorname{Tr} \int_M \mathrm{d}^2\sigma \left(\frac{1}{2\pi\mathrm{i}} \oint_c \mathrm{d}z \frac{\mathrm{e}^{-\xi z}}{z - R_n^\dagger R_n} - \frac{1}{2\pi\mathrm{i}} \oint_c \mathrm{d}z \frac{\mathrm{e}^{-\xi z}}{z - R_n R_n^\dagger} \right)_{\xi\text{-independent}}$$
$$= (2n+1)\left(\frac{-1}{4\pi} \int_M \mathrm{d}^2\sigma\, \partial^2 \ln \rho \right). \tag{5.1}$$

The bracket on the right-hand side of (5.1) can be replaced by the topological Euler characteristic $\chi(M)$ on the world sheet of string, M, an orientable compact surface without boundaries

$$\chi(M) = \frac{1}{4\pi} \int_M \mathrm{d}^2\sigma \sqrt{g} R \tag{5.2}$$

where R is the scalar curvature and

$$\sqrt{g} R = -\partial^2 \ln \rho \tag{5.3}$$

with $g_{ab} = \rho \delta_{ab}$.

On the other hand, the left-hand side of (5.1) can be interpreted as the difference between the dimension of Ker $R_n^\dagger R_n$ and that of Ker $R_n R_n^\dagger$, i.e.

$$\text{LHS} = \dim \operatorname{Ker} R_n^\dagger R_n - \dim \operatorname{Ker} R_n R_n^\dagger = \dim \operatorname{Ker} R_n - \dim \operatorname{Ker} R_n^\dagger$$

which is nothing but the analytical index of the differential operator R_n

$$\operatorname{index}(R_n) \equiv \dim \operatorname{Ker} R_n - \dim \operatorname{Ker} R_n^\dagger. \tag{5.4}$$

Thus (5.1) becomes

$$\operatorname{index}(R_n) = (2n+1)\chi(M). \tag{5.5}$$

This is just the Atiyah-Singer index theorem or its equivalent, the Riemann-Roch theorem. We have merely derived it in a special case on a two-dimensional world sheet of strings.

Actually, in the case of the conformal anomaly, life is not so easy. As one can see from (2.16) and (4.3), if we still interpret the insertion of the residue regulator (3.3) with a ξ-independent prescription as a trick for choosing the zero modes of the operator $R_n^{(i)}$ ($R_n^{(i)\dagger} = R_{-n-1}^{(i)}$), then the anomalous action is related to a summation of the conformal index $R_n^{(i)}(F)$ over the field F where

$$\text{conformal index } R_n^{(i)}(F) \equiv (-1)^F \tfrac{1}{2}(n+1) \dim \operatorname{Ker} R_n^{(i)} \tag{5.6}$$

with $\frac{1}{2}(n+1)$ being the conformal dimension of field F under the conformal transformations (2.4) and (4.2), and

$$(-1)^F = \begin{cases} 1 & \text{if } F \text{ is a boson field} \\ -1 & \text{if } F \text{ is a fermion field.} \end{cases} \tag{5.7}$$

Obviously (5.6) is quite different from the quantity index R_n defined in (5.4).

Now let us turn to the second remark. As shown in (3.14) and (3.15), within

$$Z[\rho] = \mathrm{e}^{\Delta\Gamma} Z[\rho\, \mathrm{e}^\alpha] \tag{5.8}$$

the action in $Z[\rho\, \mathrm{e}^\alpha]$ (see (2.5)) is

$$S' = \int \mathrm{d}^2\sigma \left(\tfrac{1}{4} \tilde{X}^\mu R_0^{(\alpha)\dagger} R_0^{(\alpha)} \tilde{X}_\mu + \xi R_1^{(\alpha)} \tilde{\eta} \right) \tag{5.9}$$

with

$$R_n^{(\alpha)} = \mathrm{i}\exp(\tfrac{1}{2}n\alpha)\rho^{n/2}\bar{\partial}\rho^{-(n+1)/2}\exp[-\tfrac{1}{2}(n+1)\alpha]. \tag{5.10}$$

Relabelling the variable

$$\rho\,\mathrm{e}^{\alpha} = \rho'$$

then

$$R_n^{(\alpha)} = R'_n = \mathrm{i}(\rho')^{n/2}\bar{\partial}(\rho')^{-(n+1)/2}$$

and S' is invariant under the conformal transformations

$$\begin{aligned} &\rho' \to \mathrm{e}^{\beta}\rho' && \tilde{X}^{\mu} \to \mathrm{e}^{\beta/2}\tilde{X}^{\mu} \\ &\xi \to \mathrm{e}^{-\beta/2}\xi && \tilde{\eta} \to \mathrm{e}^{\beta}\tilde{\eta}. \end{aligned} \tag{5.11}$$

But correspondingly

$$\Delta\Gamma = -\frac{(D-26)}{72\pi}\int \mathrm{d}^2\sigma\,\alpha(\sigma)(3\partial^2\ln\rho' - \partial^2\alpha) \tag{5.12}$$

is not invariant under the transformations (5.11). Moreover, there are two scalar fields, ρ' and α, involved in $\Delta\Gamma$. Notice that, however, in establishing the generating functional (2.2) by the Faddeev–Popov trick, one has already dropped an integral over the reparametrisation group manifold $\int[\mathrm{d}g]$. This is valid only for a theory without a conformal anomaly. Now for $\Delta\Gamma \neq 0$, $(D \neq 26)$, one can write down the following functional ($\rho' \to \rho$ again)

$$\tilde{Z} = \int[\mathrm{d}\alpha]Z = \int[\mathrm{d}\alpha][\mathrm{d}\sqrt{\rho}][\mathrm{d}\tilde{X}^{\mu}][\mathrm{d}\xi][\mathrm{d}\tilde{\eta}]\exp(-S+\Delta\Gamma) \tag{5.13}$$

with S defined in (2.3) and $\Delta\Gamma$ given by (5.12) ($\rho' \to \rho$).

Performing the functional integration with respect to α and neglecting the irrelevant multiplication constant, one obtains

$$\tilde{Z} = \int[\mathrm{d}\sqrt{\rho}][\mathrm{d}\tilde{X}^{\mu}][\mathrm{d}\xi][\mathrm{d}\tilde{\eta}]\exp(-S_{\mathrm{eff}}) \tag{5.14}$$

$$S_{\mathrm{eff}} = \int \mathrm{d}^2\sigma\left(-\frac{1}{2}\frac{\tilde{X}^{\mu}}{\sqrt{\rho}}\partial_a\partial_a\frac{\tilde{X}_{\mu}}{\sqrt{\rho}} + \mathrm{i}\xi\sqrt{\rho}\bar{\partial}\frac{1}{\rho}\tilde{\eta} + \frac{(D-26)}{32\pi}(\partial\ln\rho)^2\right). \tag{5.15}$$

Hence, at the cost of sacrificing the conformal invariance, a scalar metric field, $\rho(\sigma)$, becomes a dynamical field. This is the Liouville field theory but without the $\mu^2\rho$ term. Could it serve as a starting point for constructing a quantum bosonic string at subcritical dimension $(D<26)$? We do not know yet. The discussion of the spinning string is almost the same. A similar argument had been made for anomalous gauge field theories, but the outcome seems quite different (Harada and Tsutsui 1987, Falck and Kramer 1987). Nevertheless, further study is needed.

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